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Mixed convection along a vertical cone for fluids of any Prandtl number: case of constant wall temperature

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Abstract An analysis of steady laminar mixed convection boundary layer flow along a vertical cone of constant wall temperature is presented. A mixed convection parameter ξ , as proposed by Lin and Chen, is used to serve as a controlling parameter that determines the relative importance of the forced and the free convection flows. New coordinates and dependent variables are then defined in terms of ξ , so that the transformed non-similar boundary layer equations give computationally efficient numerical solutions which are valid over the entire range of mixed convection flow from the forced convection limit to the free convection limit for fluids of any Prandtl number. The effects of the mixed convection parameter ξ and the Prandtl number Pr on the velocity and temperature profiles as well as on the skin friction and heat transfer coefficients are shown for both cases of buoyancy assisting and buoyancy opposing flow conditions.

Nomenclature

$C_{\rm f}$	= local skin friction	α	= thermal
g	= acceleration due to gravity	β	= thermal
k	= thermal conductivity	η	= non-sim
Nu	= local Nusselt number	θ	= non-dim
Pr	= Prandtl number	λ	= non-dim
r	= distance from the axis of the cone to	ν	= kinemat
	the surface	ξ	= non-din
Ra	= Rayleigh number	-	mixed c
Re	= Reynolds number	ρ	= density
Т	= temperature	ψ	= stream f
u, v	= velocity components along x, y	Φ	= cone apo
	directions		_
x	= streamwise coordinate	Su	berscript
v	= coordinate normal to the surface	, 1	- different

Greek symbols

- diffusivity
- expansion coefficient
- ilarity variable
- ensional temperature
- ensional parameter
- tic viscosity
 - nensional coordinate and onvection parameter
- unction
 - ex half-angle
 - differentiation with respect to η



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HFF Introduction

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The problem of laminar free convection flow along a vertical cone has been treated in the literature by many researchers since 1953. Merk and Prins (1953) found the similarity solutions for the case of an isothermal cone whereas Hering and Grosh (1962) have obtained a number of similarity solutions for cones with prescribed wall temperatures being a power function of the distance from the apex along the generator. Further results were obtained by Hering (1965) and Sparrow and Guinle (1968) for small values of the Prandtl number (Pr) and by Roy (1974) for large values of the Pr, respectively. Also, Alamgir (1989) has investigated the overall heat transfer in laminar natural convection from vertical cones using the integral method. Further, Pop and Takhar (1991) have studied the compressibility effects in laminar free convection from a vertical cone, while Hossain and Paul (2001) and Watanabe (1991) have considered the effect of suction and injection when the cone surface is permeable. In all these papers the cone angle was large so that the effect of transverse curvature was neglected, i.e. the boundary-layer thickness is small compared to the local radius of the cone. From the mathematical point of view, such flows are similar, and, as such, the governing differential equations can be reduced to ordinary differential equations. However, Gorla and Stratman (1986) and Kuiken (1968) have considered the case of a slender cone where the transverse curvature effect is considered. The transformed boundary-layer equations are non-similar, and these equations were solved in terms of series expansion of the transverse curvature variable. Recently, Wang (1991) and Wang et al. (1994) have presented results for the free convection boundary-layer flow due to a rotating cone with constant or variable surface temperature.

A literature search reveals that relatively little work has been done on the problem of mixed convection flow along a vertical cone. To the authors' best knowledge there is only one paper by Kumari *et al.* (1989) who investigated the steady mixed convection flow over a vertical cone for two values of the Pr, namely Pr = 0.733 (air) and Pr = 6.7 (water). However, these authors have considered only the case of assisting flow.

The present analysis concentrates on the steady laminar mixed convection boundary-layer flow over a vertical isothermal cone for fluids of any Pr when both the assisting and opposing flows are considered. New variables, as proposed by Lin and Chen (1988), have been used to obtain numerical solutions of the transformed non-similar equations. These variables are uniformly valid over the entire region of mixed convection flow from pure forced convection limit to pure free convection limit, respectively. The resulting non-similarity boundary-layer equations are solved numerically using the Keller-box scheme for fluids of any Pr from very small to extremely large values $(0.001 \le Pr \le 10.000)$.

Basic equations

Consider a vertical circular cone with half angle Φ , which is placed in a large body of a viscous and incompressible fluid of ambient temperature T_{∞} flowing in the upward direction with the uniform velocity U_{∞} . The physical model and a suitable coordinate system are given in Figure 1. The coordinates x and y are measured from the cone apex along the surface and outward normal to the surface, respectively. It is assumed that the uniform surface temperature of the cone is T_{w} , where $T_{w} > T_{\infty}$ for assisting flow and $T_{w} < T_{\infty}$ for opposing flow, respectively. It is also assumed that the cone angle Φ is large, so that the transverse curvature effect is negligible. Under these assumptions along with the Boussinesq approximation, the boundary-layer equations for the steady, non-dissipative and axisymmetric laminar flow over a vertical cone are

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) = 0 \tag{1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = v \frac{\partial^2 u}{\partial y^2} \pm g\beta(T - T_\infty)\cos\phi$$
(2)

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$
(3)

where *u*, *v* are the velocity components along the *x* and *y* axes, *T* is the fluid temperature, *g* is the acceleration due to gravity, α is the thermal diffusivity, β is the coefficient of thermal expansion, ν is the kinematic viscosity. The \pm signs in equation (2) correspond to the case of assisting flow ($T_w > T_\infty$) and to the case of opposing flow ($T_w < T_\infty$), respectively. The boundary conditions of equations (1)-(3) are

$$u = v = 0, \quad T = T_{w} \quad \text{on} \quad y = 0$$

 $u \to U_{\infty}, \quad T \to T_{\infty} \quad \text{as} \quad y \to \infty$ (4)



Figure 1. Physical model and coordinate system

HFF In order to study the entire range of mixed convection flow, we introduce the following variables proposed by Lin and Chen (1988)

$$\xi = \xi(x), \quad \eta = (y/x)\lambda, \quad \psi = \alpha\lambda r(x)f(\xi,\eta), \quad \theta(\xi,\eta) = (T - T_{\infty})/|\Delta T|$$
(5)

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where $\Delta T = T_w - T_\infty$ and ψ is the stream function which is defined as

$$u = \frac{1}{r} \frac{\partial \psi}{\partial y}, \quad v = -\frac{1}{r} \frac{\partial \psi}{\partial x}$$
(6)

and r(x) is the local cone radius given by

$$r(x) = x\sin\Phi \tag{7}$$

Also, the parameters $\xi(x)$ and $\lambda(x)$ are defined as

$$\xi(x) = [1 + (\omega \operatorname{Re}_{x})^{1/2} / (\sigma \operatorname{Ra}_{x})^{1/4}]^{-1} = \zeta / (1 + \zeta),$$

$$\zeta(x) = (\sigma \operatorname{Ra}_{x})^{1/4} / (\omega \operatorname{Re}_{x})^{1/2},$$

$$\lambda(x) = (\omega \operatorname{Re}_{x})^{1/2} + (\sigma \operatorname{Ra}_{x})^{1/4} = (\omega \operatorname{Re}_{x})^{1/2} / (1 + \zeta)$$

$$= (\omega \operatorname{Re}_{x})^{1/2} / (1 + \zeta) = (\sigma \operatorname{Ra}_{x})^{1/4} / \xi = (\sigma \operatorname{Ra}_{x})^{1/4} (1 + \zeta) / \zeta$$
(8)

with

$$\sigma = \Pr/(1 + \Pr), \quad \omega = \Pr/(1 + \Pr)^{1/3}$$
 (9)

Pr being the Prandtl number. Further, Re_x is the local Reynolds number and Ra_x is the local Rayleigh number and they are defined as

$$\operatorname{Re}_{x} = U_{\infty} x / \nu, \quad \operatorname{Ra}_{x} = g\beta |\Delta T| x^{3} / \alpha \nu \tag{10}$$

It should be noted that the parameter $\zeta(x)$ maps the entire mixed convection domain from $0 \le \zeta \le \infty$ to $0 \le \xi \le 1$, respectively. In addition, the parameter ξ can serve as a controlling index that properly indicates the relative importance of the forced and free convection flow for fluids of any Prandtl number.

Substituting variables (8) into equations (1)-(3), these can be written as

$$\Pr \frac{\partial^3 f}{\partial \eta^3} + \frac{6+\xi}{4} f \frac{\partial^2 f}{\partial \eta^2} - \frac{\xi}{2} \left(\frac{\partial f}{\partial \eta}\right)^2 \pm (1+\Pr)\xi^4 \theta$$
$$= \frac{\xi(1-\xi)}{4} \left(\frac{\partial f}{\partial \eta} \frac{\partial^2 f}{\partial \xi \partial \eta} - \frac{\partial f}{\partial \xi} \frac{\partial^2 f}{\partial \eta^2}\right)$$
(11)

$$\frac{\partial^2 \theta}{\partial \eta^2} + \frac{6+\xi}{4} f \frac{\partial \theta}{\partial \eta} = \frac{\xi(1-\xi)}{4} \left(\frac{\partial f}{\partial \eta} \frac{\partial \theta}{\partial \xi} - \frac{\partial f}{\partial \xi} \frac{\partial \theta}{\partial \eta} \right)$$
(12) Mixed convection

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subject to the boundary conditions (4) which become

$$f(\xi,0) = \frac{\partial f}{\partial \eta}(\xi,0) = 0, \quad \theta(\xi,0) = 1$$

$$\frac{\partial f}{\partial \eta} \rightarrow (1 + \Pr)^{\frac{1}{3}}(1 - \xi)^{2}, \quad \theta \rightarrow 0 \quad \text{as} \quad \eta \rightarrow \infty$$
(13)

For the limiting case of pure forced convection flow ($\xi = 0$), equations (11) and (12) become:

$$\Pr f''' + \frac{3}{2}ff'' = 0 \tag{14}$$

$$\theta'' + \frac{3}{2}f \ \theta' = 0 \tag{15}$$

along with the boundary conditions:

$$f(0) = f'(0) = 0, \quad \theta(0) = 1$$
$$f' \to (1 + \Pr)^{\frac{1}{3}}, \quad \theta \to 0 \quad \text{as} \quad \eta \to \infty$$
(16)

On the other hand, for the pure free convection case ($\xi = 1$), we have:

$$\Pr f''' + \frac{7}{4}f f'' - \frac{1}{2}f'^2 \pm (1 + \Pr)\theta = 0$$
(17)

$$\theta'' + \frac{7}{4}f \ \theta' = 0 \tag{18}$$

subject to the boundary conditions

$$f(0) = f'(0) = 0, \quad \theta(0) = 1$$

$$f' \to 0, \quad \theta \to 0 \quad \text{as} \quad \eta \to \infty$$
 (19)

where primes denote differentiation with respect to η .

The physical quantities of primary interest are the reduced velocity, and temperature profiles given by

$$\frac{u}{U_{\infty}} = \frac{1}{(1+\Pr)^{1/3}(1-\xi)^2} \frac{\partial f}{\partial \eta}$$
(20)

as well as the local skin friction coefficient $C_{\rm f}$ and the local Nusselt number Nu which can be expressed as

$$C_{\rm f} {\rm Re}^{\frac{1}{2}} = \frac{\sigma^{\frac{1}{2}}}{(1-\xi)^3} \frac{\partial^2 f}{\partial \eta^2}(\xi,0)$$
(21)

$$\frac{\mathrm{Nu}}{(\omega \mathrm{Re})^{\frac{1}{2}}} = -\frac{1}{1-\xi} \frac{\partial \theta}{\partial \eta}(\xi, 0) \quad \text{or} \quad \frac{\mathrm{Nu}}{(\sigma \mathrm{Ra})^{\frac{1}{2}}} = -\frac{1}{\xi} \frac{\partial \theta}{\partial \eta}(\xi, 0) \tag{22}$$

Results and discussion

Equations (11) and (12) subject to the boundary conditions (13) have been solved numerically using the Keller-box method, which is very well described in the book by Cebeci and Bradshaw (1984). Representative results have been obtained for a wide range of Pr and values of the mixed convection parameter varying from $\xi = 0$ to 1. To verify the accuracy of the present method, the comparisons in the reduced local skin friction coefficient f''(1,0) and the reduced local heat transfer rate $-\theta'(1,0)$ have been shown in Tables I and II in the case of pure free convection ($\xi = 1$). It is seen that the present results are in

		Present results			Hering (1965)	
	Pr	f''(1,0)	$-\theta'(1,0)$	f''(1,0)	$-\theta'(1,0)$	
	0.001	47.370193	0.776212	47.96	0.7803	
	0.003	26.362112	0.766197	26.45	0.7669	
Table I.	0.01	13.530340	0.747159	13.55	0.7475	
Values of $f''(1,0)$ and	0.03	7.181065	0.718236	7.185	0.7185	
$-\theta'(1,0)$ for pure free	0.1	3.464674	0.668221	3.466	0.6683	
convection case and	0.7	0.979573	0.539140	0.9796	0.5392	
small values of Pr	1.0	0.769416	0.510383	0.7694	0.5104	
		Present results	Roy (19	974) Her	ing and Grosh (1962)	

	Pr	f''(1,0)	$-\theta'(1,0)$	f''(1,0)	$-\theta'(1,0)$	$-\theta'(1,0)$
Table II. Values of $f''(1,0)$ and $-\theta'(1,0)$ for pure free convection case and large values of Pr	0.72 1 10 100 1,000 10.000	0.815841 0.769416 0.487743 0.289660 0.166242 0.094220	0.451097 0.510383 1.033969 1.922525 3.469263 6.199769	0.859998 0.489898 0.289656 0.166139 0.094046	0.526498 1.035362 1.922875 3.469980 6.199854	0.4511

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excellent agreement with those of Hering (1965) for small values of Pr and with those of Roy (1974) for large values of Pr, respectively. We also notice that Hering and Grosh (1962) have obtained for Pr=0.72 the value $-\theta'(1,0) = 0.4511$ which is again in excellent agreement with the value obtained in the present paper.

The reduced velocity profiles,

$$\frac{\partial f}{\partial \eta}(\xi,\eta)$$

are shown in Figures 2-6 for different values of the Pr and of the streamwise coordinate (or mixed convection parameter) ξ . Figures 2 and 3 show clearly the evolution of these profiles from pure forced convection limit ($\xi = 0$) to pure free convection limit ($\xi = 1$) in both the cases of assisting and opposing flow, respectively. It is also seen that these velocity profiles are smaller for air (Pr = 0.7) than for water (Pr = 6.8). In addition, we can see, as expected, that





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Figure 3.

Reduced velocity profiles, $\partial f/\partial \eta (\xi, \eta)$, in the case of opposing flow for some values of ξ varying between $\xi = 0$ (forced convection) and $\xi = 1$ (free convection). (a) Pr = 0.72 (air), and (b) Pr = 6.8 (water)

Figure 4. Effect of Pr on the non-dimensional velocity profiles, u/U_{∞} , as given by equation (20) in the case of forced convection flow ($\xi = 0$) for Pr = 0.001, 0.01, 0.1, 1, 10, 100 and 1,000



(y/x) Re^{1/2}



there is a rather substantial reverse flow in the case of opposing flow. Further, Figures 4-6 show the development of the non-dimensional velocity profiles, u/U_{∞} , as given by equation (20), for both the assisting and opposing flow cases. Figure 4 shows that for pure forced convection case ($\xi=0$) the velocity profiles, as expected (Bejan, 1995), decrease with an increase of the Pr. However, in the case of assisting flow, the velocity profiles u/U_{∞} decrease with the increase of the mixed convection parameter ξ in the range $0 \le \xi \le 0.4$ and increase for $\xi \ge 0.8$. However, these profiles decrease with an increase of ξ in the case of opposing flow, as can be seen from Figure 6. For $\xi \ge 0.8$, the flow is completely reversed.

The evolution of the reduced temperature profiles, $\theta(\xi, \eta)$, are shown in Figures 7-9 for both the cases of assisting and opposing flow conditions, and for different values of the parameters ξ and Pr. We can see that these profiles are very similar. However, they increase with an increase of the values of the parameter.



The variation of the reduced skin friction coefficient,

$$\frac{\partial^2 f}{\partial \eta^2}(\xi,0),$$

with ξ and Pr in the range $0.001 \le Pr \le 10.000$ is shown in Figure 10 for both the cases of buoyancy assisting and buoyancy opposing flow conditions. From Figure 10(a), we see that in the case of assisting flow the reduced skin friction decreases to a minimum value near $\xi = 0.4$ for very small values of Pr. However, for very large values of Pr, the reduced skin friction varies almost linearly. On the other hand, Figure 9(b) shows that in the case of buoyancy opposing flow the boundary-layer separates and the separation is delayed as Pr increases.



Finally, Figures 11 and 12 show the variation of the reduced heat transfer rate,

$$-\frac{\partial \theta}{\partial \eta}(\xi,0)$$

with ξ or Pr. It can be seen from Figure 11(a) that in the case of assisting flow the reduced wall heat transfer attains minimum values for ξ in the range $0.4 \le \xi \le 0.6$ when $\Pr \le 100$. These minimum values increase with increasing Pr. However, for values of ξ up to about $\xi = 0.5$, the reduced heat transfer rate is almost constant for all the values of Pr considered. Further, Figure 12 shows that in the case of opposing flow, the reduced heat transfer rate varies almost linearly for all small values of Pr, while it oscillates as $\Pr \ge 1$. However, the values of the rate of heat transfer gets higher as Pr increases, but decreases with the parameter ξ . HFF 13,7

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Figure 8.

Reduced temperature profiles, $\theta(\xi, \eta)$, in the case of buoyancy opposing flow for some values of ξ varying between $\xi = 0$ (forced convection) and $\xi = 1$ (free convection). (a) Pr = 0.72 (air), and (b) Pr = 6.8 (water)









Conclusion

In this paper, we have presented a boundary-layer analysis for the steady mixed convection flow along an isothermal vertical cone. A stretched streamwise coordinate ξ and a pseudo-similarity variable η , as proposed by Lin and Chen (1988), have been used to yield computationally efficient numerical solutions that are valid over the entire range of mixed convection flow from the pure forced convection limit ($\xi = 0$) to the pure free convection limit ($\xi = 1$) for fluids of any Pr between 0.001 and 10.000. The results are given for the velocity and temperature distributions, as well as for the reduced skin friction coefficient and the reduced heat transfer rate for various values of the mixed convection parameter ξ and Pr. It was found that for some specific values of these parameters, the present results agree very well with those reported by Hering (1965), Hering and Grosh (1962) and Roy (1974). It is also shown that in the case of opposing flow there is a reversed flow for some values of the mixed convection parameter.

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Figure 11. Variation of the reduced heat transfer, $-\partial \theta / \partial \eta (\xi, 0)$, given by equation (22) with ξ varying between $\xi = 0$ (forced convection) and $\xi = 1$ (free convection) for Pr = 0.001, 0.01, 0.1, 1, 10, 100, 1,000 and 10,000 in the cases of assisting flow and opposing flow, respectively







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